M346 Second Midterm Exam, November 9, 2004

1. Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors of A.

b) Write down the most general solution to the difference equation $\mathbf{x}(n) = A\mathbf{x}(n-1)$.

c) Find $\mathbf{x}(n)$ when $\mathbf{x}(0) = (7,1)^T$.

2. Let $A = \begin{pmatrix} 3 & 2 \\ -5 & -4 \end{pmatrix}$ and consider the differential equation $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.

a) Find the eigenvalues and eigenvectors of A.

b) How many stable modes are there, and what are they? How many unstable modes are there and what are they? How many neutrally stable modes are there and what are they? What is the dominant mode?

c) If $\mathbf{x}(0) = (\pi, \sqrt{17})^T$, find the limit, as $t \to \infty$, of $x_1(t)/x_2(t)$.

3. Consider the NONLINEAR system of differential equations

$$\frac{dx_1}{dt} = 2(x_2^2 - 1)$$
$$\frac{dx_2}{dt} = (x_1^2 - 1)/2$$

a) Find the fixed points (there are four of them).

b) For each fixed point, approximate the deviations from that fixed point by a linear system of equations.

c) Which fixed points are stable? Unstable? Borderline?

4. In $\mathbb{R}^3$, with the usual inner product, consider the vectors $\mathbf{b}_1 = (1,2,2)^T$, $\mathbf{b}_2 = (4,4,3)^T$, $\mathbf{b}_3 = (2,9,-1)^T$.

a) Use the Gram-Schmidt process to convert this basis for $\mathbb{R}^3$ into an orthogonal basis for $\mathbb{R}^3$.

b) If $\mathbf{v} = (5,2,9)^T$, compute $P_{\mathbf{b}_1}\mathbf{v}$.

5. On $\mathbb{R}^3$, let $\mathbf{b}_1 = (3,4,5)^T$, $\mathbf{b}_2 = (5,-10,5)^T$, $\mathbf{b}_3 = (7,1,-5)^T$. Note that these vectors are orthogonal. Let $\mathbf{v} = (2,1,2)^T$.

a) Write $\mathbf{v}$ as a linear combination of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$.

b) Let M be a matrix with eigenvalues $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 15$ and eigenvectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. Compute $M\mathbf{v}$.