M346 First Midterm Exam Solutions, September 21, 2004

1. Let V be the subspace of $\mathbb{R}^4$ defined by the equation $x_1+x_2+x_3+x_4=0$.

a) Find the dimension of V.

V is the null space of the rank-1 matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$, and so has dimension 4-1=3.

b) Find a basis for V. [Any basis will do, but the simpler your answer, the easier part (c) will be. Be sure that each of your vectors really is in V, and that they are linearly independent]

Treat x_1 as the constrained variable and the others as free: $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

c) Let $L(\mathbf{x}) = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{pmatrix}$. Note that L takes V to V, and can be viewed as an operator *on* V. Find the matrix $[L]_{\mathcal{B}}$, where $\mathcal{B}$ is the basis you found in part (b).

Since $L(\mathbf{b}_1) = -\mathbf{b}_3$, $L(\mathbf{b}_2) = \mathbf{b}_3 - \mathbf{b}_1$ and $L(\mathbf{b}_3) = -\mathbf{b}_1$, the matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$. Of course, if you picked a different basis in (b) then you would get a different matrix in (c).

2. In $\mathbb{R}^2$, consider the basis $\mathbf{b}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

a) Find the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$, where $\mathcal{E}$ is the standard basis.

$P_{\mathcal{E}\mathcal{B}} = ([\mathbf{b}_1]_{\mathcal{E}}[\mathbf{b}_2]_{\mathcal{E}}) = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$. $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$.

b) If $\mathbf{v} = \begin{pmatrix} 13 \\ -2 \end{pmatrix}$, find $[\mathbf{v}]_{\mathcal{B}}$.

$[\mathbf{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{v}]_{\mathcal{E}} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 13 \\ -2 \end{pmatrix} = \begin{pmatrix} 32 \\ -49 \end{pmatrix}$. You can check that $\mathbf{v}$ is indeed equal to $32\mathbf{b}_1 - 49\mathbf{b}_2$.

c) Let $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_1 + x_2 \end{pmatrix}$. Find $[L]_{\mathcal{E}}$ and $[L]_{\mathcal{B}}$. By inspection, $[L]_{\mathcal{E}} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$. Then we compute $[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[L]_{\mathcal{E}}P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} -12 & -7 \\ 22 & 13 \end{pmatrix}$.

3. Consider the coupled first-order differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 + 2x_2 \\ \frac{dx_2}{dt} &= 2x_1 + x_2 \end{aligned}$$

Define the new variables $y_1(t) = x_1(t) + x_2(t)$, $y_2(t) = x_1(t) - x_2(t)$.

a) Rewrite the system of equations completely in terms of y_1 and y_2. (That is, express dy_1/dt and dy_2/dt as functions of y_1 and y_2.)

$dy_1/dt = 3y_1$, $dy_2/dt = -y_2$. This implies that $y_1(t) = e^{3t}y_1(0)$ and $y_2(t) = e^{-t}y_2(0)$.

b) Given the initial conditions $x_1(0) = 1$, $x_2(0) = 0$, find $x_1(t)$ and $x_2(t)$.

$y_1(0) = 1 + 0 = 1$ and $y_2(0) = 1 - 0 = 1$, so $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$, so $x_1(t) = (e^{3t} + e^{-t})/2$ and $x_2(t) = (e^{3t} - e^{-t})/2$.

4. Let $V = \mathbb{R}_3[t]$, and let $L : V \to V$ be defined by $L(\mathbf{p})(t) = \mathbf{p}'(t) + 2\mathbf{p}''(t)$.

a) Find $[L]_{\mathcal{E}}$, where $\mathcal{E} = \{1, t, t^2, t^3\}$ is the standard basis.

By taking derivatives, we find that $L(\mathbf{b}_1) = 0$, $L(\mathbf{b}_2) = 1 = \mathbf{b}_1$, $L(\mathbf{b}_3) = 2t + 4 = 4\mathbf{b}_1 + 2\mathbf{b}_2$, $L(\mathbf{b}_4) = 3t^2 + 12t = 12\mathbf{b}_2 + 3\mathbf{b}_3$, so the matrix of L is $\begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

b) What is the dimension of the kernel of L? What is the dimension of the range of L?

Since the matrix has 3 pivots, L has rank 3, so the kernel has dimension $4 - 3 = 1$ and the range has dimension 3.

c) Find a basis for the kernel of L.

$\{1\} = \{\mathbf{b}_1\}$.

d) Find a basis for the range of L. There are many correct answers. One is

$\{1, t, t^2\}$. Another, more directly from the columns of $[L]_{\mathcal{E}}$ is $\{1, 4+2t, 12+3t^2\}$. Note that the basis vectors are elements of V (that is, functions), not columns of numbers. Their COORDINATES are columns of numbers, and form a basis for the column space of $[L]_{\mathcal{E}}$.

5. True of False? Each question is worth 4 points. You do NOT need to justify your answers, and partial credit will NOT be given.

For (a) and (b), suppose that a 4×4 matrix A row-reduces to $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

a) The null space of A is the span of $(-2, -1, 1, 0)^T$.

TRUE. x_3 is the only free variable, $x_1 = -2x_3$, $x_2 = -x_3$, and $x_4 = 0$.

b) The column space of A is the span of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

FALSE. The column space is the span of the first, 2nd and 4th columns of A, which could be almost anything.

For (c) and (d), suppose that $L : \mathbb{R}_2[t] \to M_{2,2}$ is a linear transformation, and that $B = [L]_{\mathcal{E}\mathcal{E}}$ is the matrix of L relative to the standard bases for $\mathbb{R}_2[t]$ and $M_{2,2}$.

c) If B row-reduces to something with 3 pivots, then L is 1–1.

TRUE. In that case L would have rank 3. Since $R_2[t]$ is 3-dimensional, that would make the kernel 0-dimensional, so L is 1–1.

d) If $\begin{pmatrix} 1 & 3 \\ 4 & 7 \end{pmatrix}$ is in the range of L, then $\begin{pmatrix} 1 \\ 3 \\ 4 \\ 7 \end{pmatrix}$ is in the column space of B.

TRUE. The coordinates of a vector in the range give a vector in the column space of the matrix.

e) $\mathbb{R}^3$ is the internal direct sum of the x_1-x_2 and x_1-x_3 planes.

FALSE. Those two subspaces overlap on the x_1 axis.