M346 Third Midterm Exam, April 24, 2009

1. **Is it fixed yet?** Consider the system of nonlinear differential equations:

$$\frac{dx_1}{dt} = x_1(3 - 2x_1 - x_2)$$
$$\frac{dx_2}{dt} = x_2(5 - 2x_1 - 3x_2).$$

a) Find all the fixed points.

(0,0), $(0, \frac{5}{3})$, $(\frac{3}{2}, 0)$ and (1,1)

b) For each fixed point, indicate how many stable modes, and how many unstable modes, there are.

Our derivative matrix is $\begin{pmatrix} 3 - 4x_1 - x_2 & -x_1 \\ -3x_2 & 5 - 5x_2 - 3x_1 \end{pmatrix}$

At (0,0) this is $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ which has two positive eigenvalues, so there are two unstable modes and no stable modes. This is a source.

At $(0, \frac{5}{3})$ this is $\begin{pmatrix} \frac{4}{3} & 0 \\ -5 & -5 \end{pmatrix}$ which has one positive eigenvalue, so there is one unstable mode and one stable mode. This is a saddle.

At $(\frac{3}{2}, 0)$ this is $\begin{pmatrix} -3 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ which has one positive eigenvalue, so there is one unstable mode and one stable mode. This is a saddle.

At (1,1) this is $\begin{pmatrix} -2 & -1 \\ -3 & -4 \end{pmatrix}$ which has two negative eigenvalues (-5 and -1), so there are two stable modes and no unstable modes. This is a sink.

2. **Gram crackers.** In $\mathbb{R}^3$, with the standard inner product, consider the vectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 16 \\ 9 \\ -2 \end{pmatrix}$ that form a basis for $\mathbb{R}^3$.

a) Use the Gram-Schmidt process to convert this basis to an orthogonal basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$. (Note: the vectors $\mathbf{y}_i$ do not have to be orthonormal, just orthogonal.)

$\mathbf{y}_1 = \mathbf{x}_1 = (1, 2, 3)^T$. $\mathbf{y}_2 = \mathbf{x}_2 - \frac{14}{14}\mathbf{y}_1 = (4, -1, 4)^T$. $\mathbf{y}_3 = \mathbf{x}_3 - \frac{28}{14}\mathbf{y}_1 - \frac{19}{19}\mathbf{y}_2 = (11, 8, -9)^T$.

b) The vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ can be expressed as $c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3$. Find c_1, c_2 and c_3.

Using $c_i = \langle \mathbf{y}_i | (1,0,0)^T \rangle / \langle \mathbf{y}_i | \mathbf{y}_i \rangle$ we get $c_1 = 1/14$, $c_2 = 3/19$, and $c_3 = 11/284$.

3. **When least is best.** Find **all** least-squares solutions to the system of equations

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 1 \\ 3x_1 + 6x_2 &= 1 \\ 4x_1 + 8x_2 &= 1 \end{aligned}$$

Since $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, we compute $A^T A = \begin{pmatrix} 30 & 60 \\ 60 & 120 \end{pmatrix}$ and $A^T \mathbf{b} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$. Note that $A^T A$ is singular. Solving $A^T A \mathbf{x} = A^T \mathbf{b}$ gives infinitely many solutions: $\mathbf{x} = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, where t is arbitrary.

4. **Working 24/7.** Consider the Hermitian matrix $H = \begin{pmatrix} 24 & 7 \\ 7 & -24 \end{pmatrix}$.

a) Find the eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\mathbf{b}_1$ and $\mathbf{b}_2$ of H.

The trace is zero and the determinant is $-(25)^2$, so the eigenvalues are 25 and -25, with eigenvectors $(7,1)^T$ and $(1,-7)^T$.

b) Decompose $\mathbf{x}_0 = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.

The eigenvectors are orthogonal, so the coefficients are $\langle \mathbf{b}_1 | \mathbf{x}_0 \rangle / \langle \mathbf{b}_1 | \mathbf{b}_1 \rangle = \frac{100}{50} = 2$ and $\langle \mathbf{b}_2 | \mathbf{x}_0 \rangle / \langle \mathbf{b}_2 | \mathbf{b}_2 \rangle = \frac{-50}{50} = -1$, so $\mathbf{x}_0 = 2\mathbf{b}_1 - \mathbf{b}_2$.

c) If $d\mathbf{x}/dt = H\mathbf{x}$ and $\mathbf{x}(0) = \mathbf{x}_0$ as in (b), what is $\mathbf{x}(t)$?

$\mathbf{x}(t) = 2e^{25t}\mathbf{b}_1 - e^{-25t}\mathbf{b}_2$.